# Growth

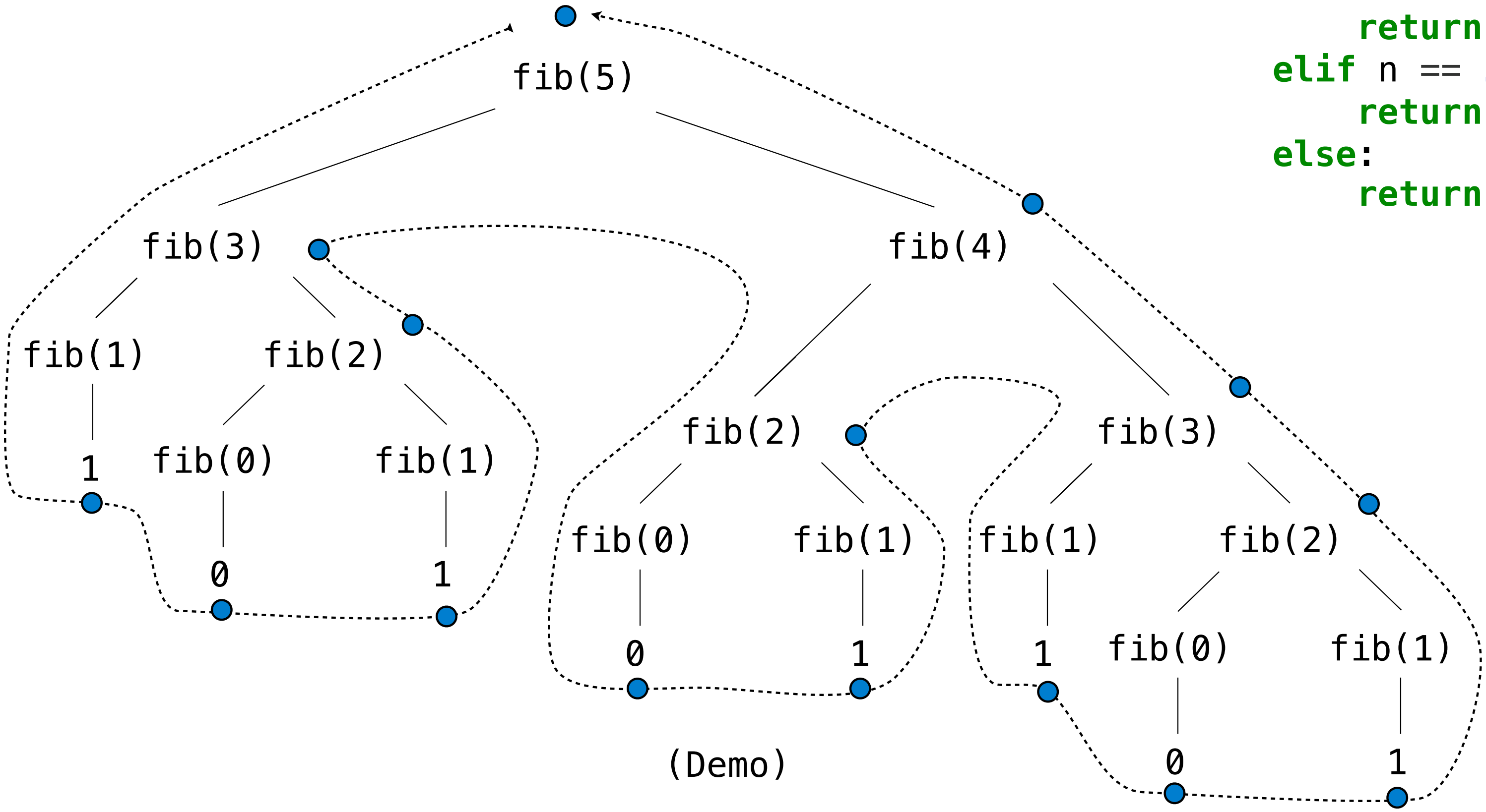
# Announcements

# Measuring Efficiency

# Recursive Computation of the Fibonacci Sequence

Our first example of tree recursion:

```
def fib(n):
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return fib(n-2) + fib(n-1)
```




(Demo)

http://en.wikipedia.org/wiki/File:Fibonacci.jpg

# Memoization

# Memoization

**Idea:** Remember the results that have been computed before

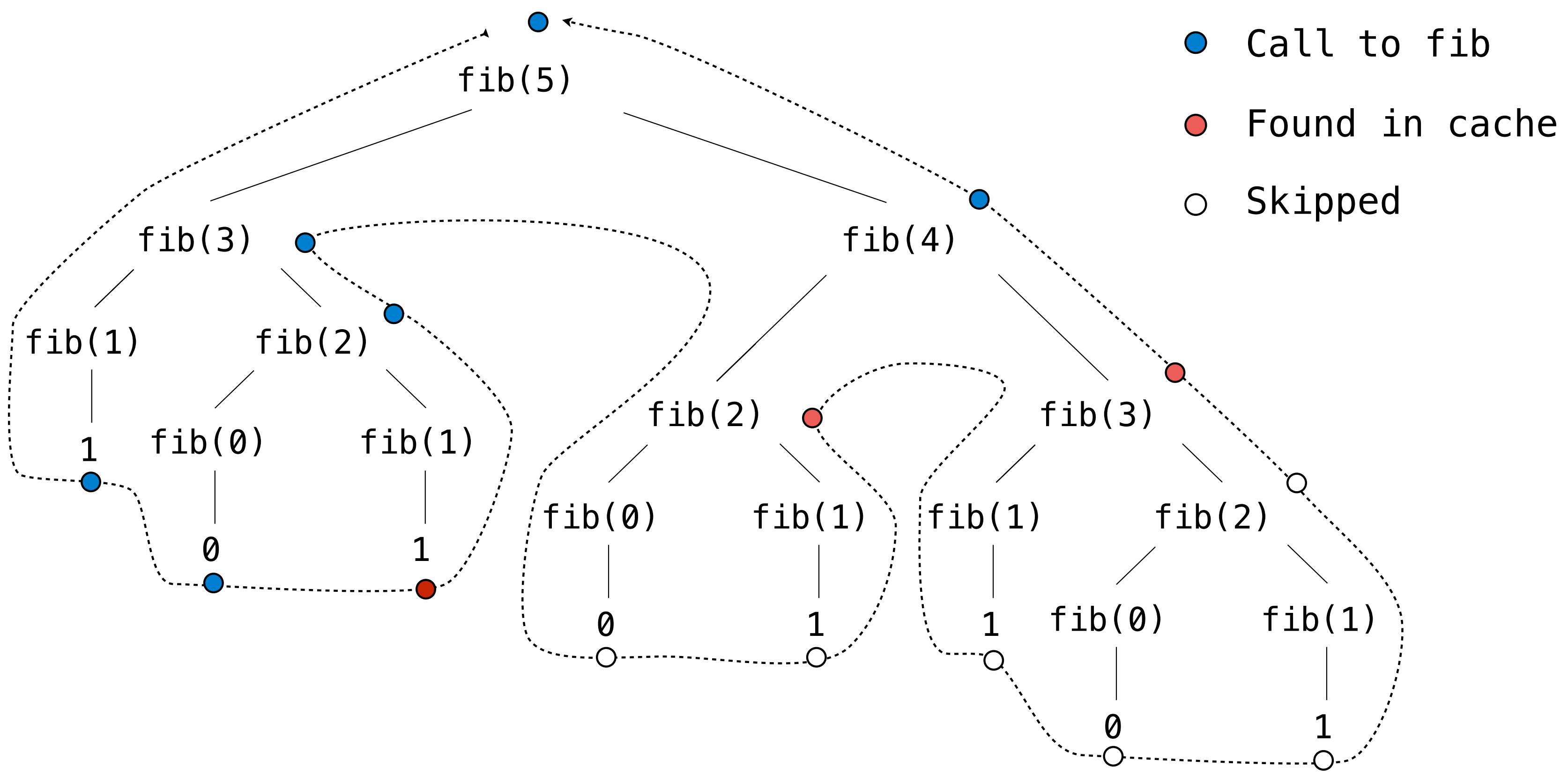
```
def memo(f):
    cache = {}
    def memoized(n):
        if n not in cache:
            cache[n] = f(n)
        return cache[n]
    return memoized
```

Keys are arguments that map to return values

Same behavior as f, if f is a pure function

(Demo)

# Memoized Tree Recursion



Space

# The Consumption of Space

Which environment frames do we need to keep during evaluation?

At any moment there is a set of active environments

Values and frames in active environments consume memory

Memory that is used for other values and frames can be recycled

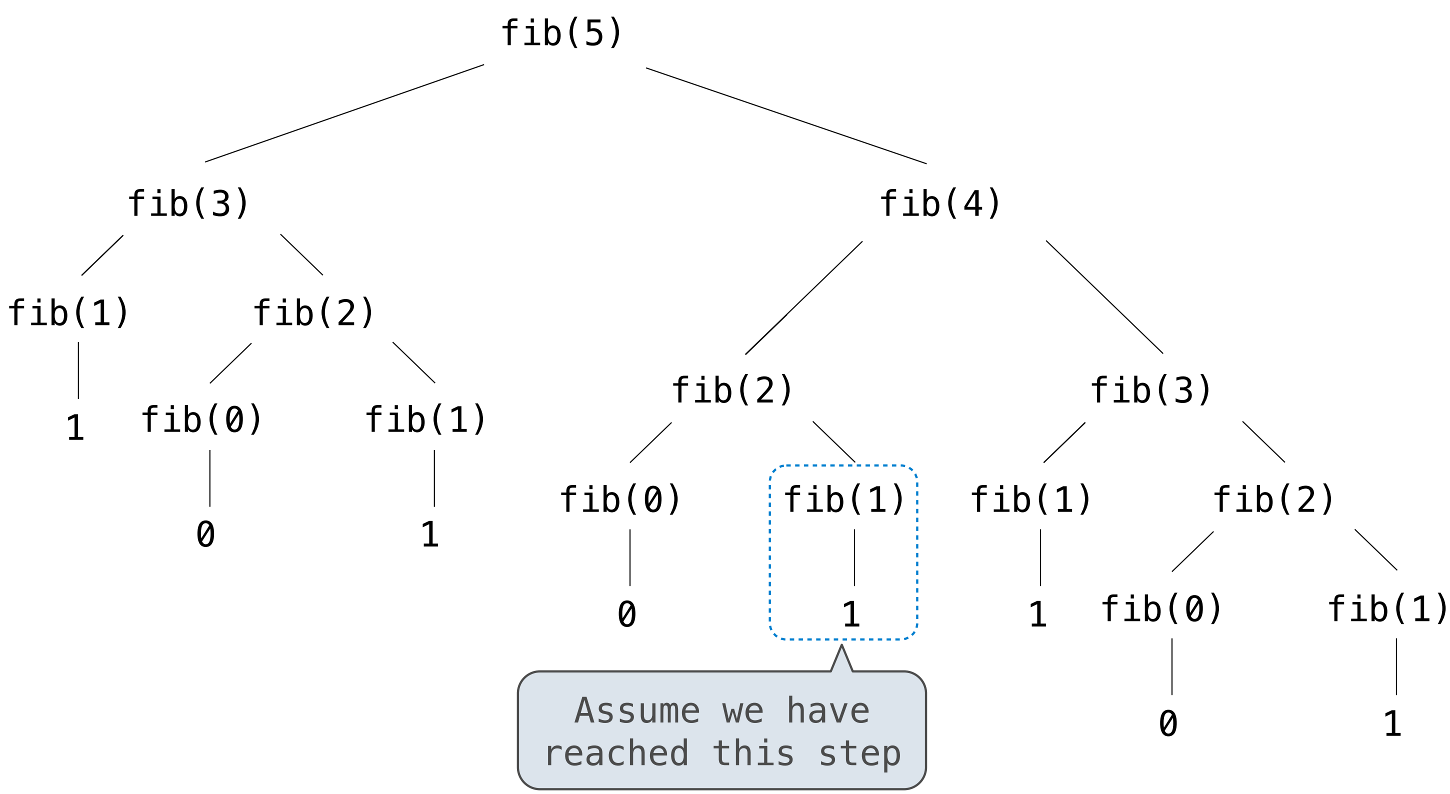
**Active environments:**

- Environments for any function calls currently being evaluated
- Parent environments of functions named in active environments

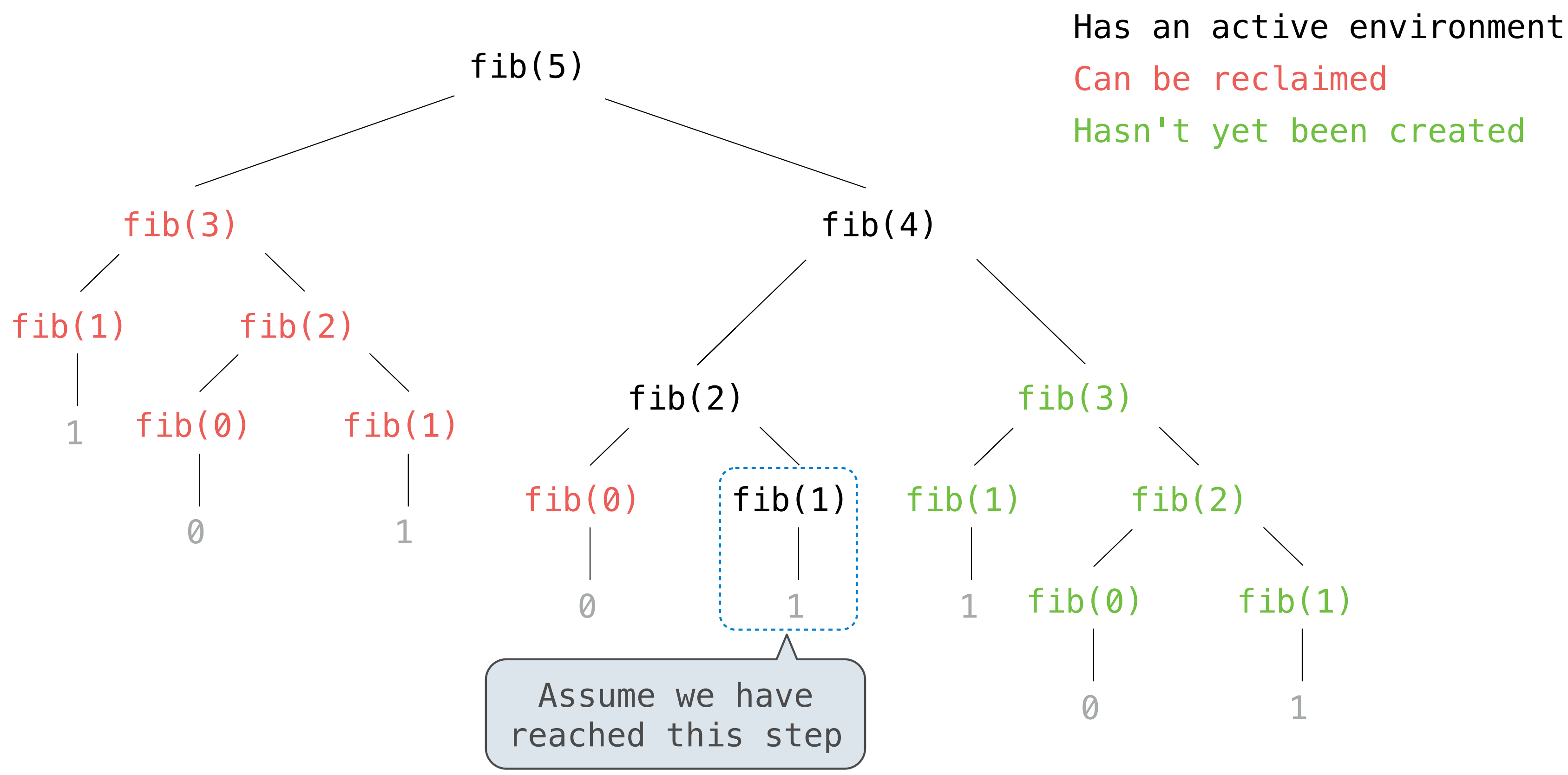
(Demo)

Interactive Diagram

# Fibonacci Space Consumption



# Fibonacci Space Consumption



# Time

# Comparing Implementations

Implementations of the same functional abstraction can require different resources

**Problem: How many factors does a positive integer n have?**

A factor k of n is a positive integer that evenly divides n

```
def factors(n):
```

| | **Time (number of divisions)** |
|---|---|
| **Slow:** Test each k from 1 through n | $n$ |
| **Fast:** Test each k from 1 to square root n<br>For every k, n/k is also a factor! | Greatest integer less than $\sqrt{n}$ |

**Question:** How many time does each implementation use division? (Demo)

# Orders of Growth

# Order of Growth

A method for bounding the resources used by a function by the "size" of a problem

**n:** size of the problem

**R(n):** measurement of some resource used (time or space)

$$R(n) = \Theta(f(n))$$

means that there are positive constants **$k_1$** and **$k_2$** such that

$$\boxed{k_1 \cdot f(n) \leq \boxed{R(n)} \leq k_2 \cdot f(n)}$$

for all **n** larger than some minimum **m**

# Order of Growth of Counting Factors

Implementations of the same functional abstraction can require different amounts of time

**Problem: How many factors does a positive integer n have?**

A factor k of n is a positive integer that evenly divides n

```
def factors(n):
```

| | Time | Space |
|---|---|---|
| **Slow:** Test each k from 1 through n | $\Theta(n)$ | $\Theta(1)$ |
| **Fast:** Test each k from 1 to square root n<br>For every k, n/k is also a factor! | $\Theta(\sqrt{n})$ | $\Theta(1)$ |

Assumption: integers occupy a fixed amount of space

(Demo)

# Exponentiation

# Exponentiation

**Goal:** one more multiplication lets us double the problem size

```
def exp(b, n):
    if n == 0:
        return 1
    else:
        return b * exp(b, n-1)
```

$$b^n = \begin{cases} 1 & \text{if } n = 0 \\ b \cdot b^{n-1} & \text{otherwise} \end{cases}$$

```
def square(x):
    return x*x

def exp_fast(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)
```

$$b^n = \begin{cases} 1 & \text{if } n = 0 \\ (b^{\frac{1}{2}n})^2 & \text{if } n \text{ is even} \\ b \cdot b^{n-1} & \text{if } n \text{ is odd} \end{cases}$$

(Demo)

# Exponentiation

**Goal:** one more multiplication lets us double the problem size

```
def exp(b, n):
    if n == 0:
        return 1
    else:
        return b * exp(b, n-1)
```

```
def square(x):
    return x*x

def exp_fast(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)
```

| | Time | Space |
|---|---|---|
| `exp` | $\Theta(n)$ | $\Theta(n)$ |
| `exp_fast` | $\Theta(\log n)$ | $\Theta(\log n)$ |

# Comparing Orders of Growth

## Properties of Orders of Growth

**Constants:** Constant terms do not affect the order of growth of a process

$$\Theta(n) \qquad \Theta(500 \cdot n) \qquad \Theta(\frac{1}{500} \cdot n)$$

**Logarithms:** The base of a logarithm does not affect the order of growth of a process

$$\Theta(\log_2 n) \qquad \Theta(\log_{10} n) \qquad \Theta(\ln n)$$

**Nesting:** When an inner process is repeated for each step in an outer process, multiply the steps in the outer and inner processes to find the total number of steps

```
def overlap(a, b):
    count = 0
    for item in a:          Outer: length of a
        if item in b:       Inner: length of b
            count += 1
    return count
```

If a and b are both length **n**, then overlap takes $\Theta(n^2)$ steps

**Lower-order terms:** The fastest-growing part of the computation dominates the total

$$\Theta(n^2) \qquad \Theta(n^2 + n) \qquad \Theta(n^2 + 500 \cdot n + \log_2 n + 1000)$$

# Comparing orders of growth (n is the problem size)

$\Theta(b^n)$ Exponential growth. Recursive fib takes

$\Theta(\phi^n)$ steps, where $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61828$

Incrementing the problem scales R(n) by a factor

$\Theta(n^2)$ Quadratic growth. E.g., overlap

Incrementing n increases R(n) by the problem size n

$\Theta(n)$ Linear growth. E.g., slow factors or exp

$\Theta(\sqrt{n})$ Square root growth. E.g., factors_fast

$\Theta(\log n)$ Logarithmic growth. E.g., exp_fast

Doubling the problem only increments R(n).

$\Theta(1)$ Constant. The problem size doesn't matter